Physics 513, Quantum Field Theory

Homework 6

Due Tuesday, 21st October 2003

Jacob Lewis Bourjaily

1. For the following derivations it will be helpful to recall the following:

$$\begin{aligned}\mathcal{P}\psi(t,\vec{x})\mathcal{P}^\dagger &= \eta_a\gamma^0\psi(t,-\vec{x});\\ \mathcal{P}\bar{\psi}(t,\vec{x})\mathcal{P}^\dagger &= \eta_a^*\bar{\psi}(t,-\vec{x})\gamma^0;\\ \mathcal{C}\psi\mathcal{C}^\dagger &= -i(\bar{\psi}\gamma^0\gamma^2)^\intercal;\\ \mathcal{C}\bar{\psi}\mathcal{C}^\dagger &= -i(\gamma^0\gamma^2\psi)^\intercal;\\ \mathcal{T}\psi(t,\vec{x})\mathcal{T}^\dagger &= \gamma^1\gamma^3\psi(-t,\vec{x});\\ \mathcal{T}\bar{\psi}(t,\vec{x})\mathcal{T}^\dagger &= -\bar{\psi}(-t,\vec{x})\gamma^1\gamma^3.\end{aligned}$$

a) We are to verify the transformation properties of $A^\mu \equiv \bar{\psi}\gamma^\mu\gamma^5\psi$ and $T^{\mu\nu} \equiv \bar{\psi}\sigma^{\mu\nu}\psi$ under $\mathcal{P}$.

Let us first consider the axial vector A^μ.

$$\begin{aligned}\mathcal{P}A^\mu\mathcal{P}^\dagger = \mathcal{P}\bar{\psi}\gamma^\mu\gamma^5\psi\mathcal{P}^\dagger &= \eta_a^*\bar{\psi}\gamma^0\gamma^\mu\gamma^5\eta_a\gamma^0\psi,\\ &= \bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\psi,\\ &= -\bar{\psi}\gamma^0\gamma^\mu\gamma^0\gamma^5\psi,\\ &= -\bar{\psi}\gamma_\mu\gamma^5\psi = -A_\mu.\end{aligned}$$

The last step can be seen by noticing that

$$\gamma^0\gamma^\mu\gamma^0 = \left\{\begin{array}{cc}\gamma^\mu & \mu=0\\ -\gamma^\mu & \mu=1,2,3\end{array}\right\} = \gamma_\mu.$$

Now we will consider the transformation of the tensor $T^{\mu\nu}$.

$$\begin{aligned}\mathcal{P}T^{\mu\nu}\mathcal{P}^\dagger = \mathcal{P}\bar{\psi}\sigma^{\mu\nu}\psi\mathcal{P}^\dagger &= \eta_a^*\bar{\psi}\gamma^0\sigma^{\mu\nu}\eta_a\gamma^0\psi,\\ &= \bar{\psi}\gamma^0\sigma^{\mu\nu}\gamma^0\psi,\\ &= \bar{\psi}\sigma_{\mu\nu}\psi = T_{\mu\nu}.\end{aligned}$$

Similar to the axial vector case, the last step is a result of directly verifying the identity

$$\gamma^0\sigma^{\mu\nu}\gamma^0 = \frac{i}{2}(\gamma^0\gamma^\mu\gamma^\nu\gamma^0 - \gamma^0\gamma^\nu\gamma^\mu\gamma^0) = \left\{\begin{array}{cc}\sigma^{\mu\nu} & \mu,\nu\neq 0 \text{ or } \mu,\nu=0\\ -\sigma^{\mu\nu} & \text{one of } \mu \text{ or } \nu = 0\end{array}\right\} = \sigma_{\mu\nu}.$$

ὅπερ ἔδει δεῖξαι

b) We are to verify the transformation properties of $V^\mu \equiv \bar{\psi}\gamma^\mu\psi$ and $A^\mu \equiv \bar{\psi}\gamma^\mu\gamma^5\psi$ under $\mathcal{C}$.

Let us first consider the transformation of the vector V^μ.

$$\begin{aligned}\mathcal{C}V^\mu\mathcal{C}^\dagger = \mathcal{C}\bar{\psi}\gamma^\mu\psi\mathcal{C}^\dagger &= -i(\gamma^0\gamma^2)^\intercal\gamma^\mu(-i)(\bar{\psi}\gamma^0\gamma^2)^\intercal,\\ &= -\bar{\psi}\gamma^0\gamma^2\gamma^{\mu\intercal}\gamma^0\gamma^2\psi,\\ &= \bar{\psi}\gamma^0\gamma^2\gamma^{\mu\intercal}\gamma^2\gamma^0\psi,\\ &= -\bar{\psi}\gamma^\mu\psi = -V^\mu.\end{aligned}$$

Let us now consider the axial vector A^μ.

$$\begin{aligned}\mathcal{C}A^\mu\mathcal{C}^\dagger = \mathcal{C}\bar{\psi}\gamma^\mu\gamma^5\psi\mathcal{C}^\dagger &= -i(\gamma^0\gamma^2\psi)^\intercal\gamma^\mu\gamma^5(-i)(\bar{\psi}\gamma^0\gamma^2)^\intercal,\\ &= -\bar{\psi}\gamma^0\gamma^2\gamma^5\gamma^{\mu\intercal}\gamma^0\gamma^2\psi,\\ &= \bar{\psi}\gamma^0\gamma^2\gamma^{\mu\intercal}\gamma^5\gamma^0\gamma^2\psi,\\ &= -\bar{\psi}\gamma^0\gamma^2\gamma^{\mu\intercal}\gamma^5\gamma^2\gamma^0\psi,\\ &= -\bar{\psi}\gamma^0\gamma^2\gamma^{\mu\intercal}\gamma^2\gamma^0\gamma^5\psi,\\ &= \bar{\psi}\gamma^\mu\gamma^5\psi = A^\mu.\end{aligned}$$

ὅπερ ἔδει δεῖξαι

c) We are to confirm the transformation properties of $P \equiv i\bar{\psi}\gamma^5\psi$ and $V^\mu \equiv \bar{\psi}\gamma^\mu\psi$ under $\mathcal{T}$.

First let us consider the transformation of the pseudo-scalar P.

$$\begin{aligned}\mathcal{T}P\mathcal{T}^\dagger = \mathcal{T}i\bar{\psi}\gamma^5\psi\mathcal{T}^\dagger &= -i(-\bar{\psi}\gamma^1\gamma^3)\gamma^5(\gamma^1\gamma^3\psi),\\ &= i\bar{\psi}\gamma^1\gamma^3\gamma^5\gamma^1\gamma^3\psi,\\ &= -i\bar{\psi}\gamma^5\psi = -P.\end{aligned}$$

Let us now consider the transformation of the vector V^μ.

$$\begin{aligned}\mathcal{T}V^\mu\mathcal{T}^\dagger = \mathcal{T}\bar{\psi}\gamma^\mu\psi\mathcal{T}^\dagger &= \bar{\psi}\gamma^3\gamma^1\gamma^{\mu*}\gamma^1\gamma^3\psi,\\ &= \bar{\psi}\gamma_\mu\psi = V_\mu.\end{aligned}$$

ὅπερ ἔδει δεῖξαι

2. a) We are to demonstrate the transformation properties of V^μ and A^μ, as previously defined, under $\mathcal{CP}$.

We have almost computed every detail necessary for our solution in question (1) above. The only transformation that we have not yet confirmed is the transformation of the vector V^μ under $\mathcal{P}$. Let us compute that now.

$$\begin{aligned}\mathcal{P}V^\mu\mathcal{P}^\dagger = \mathcal{P}\bar{\psi}\gamma^\mu\psi\mathcal{P}^\dagger &= \eta_a^*\bar{\psi}\gamma^0\gamma^\mu\eta_a\gamma^0\psi,\\ &= \bar{\psi}\gamma^0\gamma^\mu\gamma^0\psi,\\ &= \bar{\psi}\gamma_\mu\psi = V_\mu.\end{aligned}$$

By simply applying our transformation properties derived above in succession, we observe that,

$$\begin{aligned}V^\mu = \bar{\psi}\gamma^\mu\psi \quad &\xrightarrow{\mathcal{P}} \quad \bar{\psi}\gamma_\mu\psi \quad \xrightarrow{\mathcal{C}} \quad -\bar{\psi}\gamma_\mu\psi = -V_\mu\\ A^\mu = \bar{\psi}\gamma^\mu\gamma^0\psi \quad &\xrightarrow{\mathcal{P}} \quad -\bar{\psi}\gamma_\mu\gamma^5\psi \quad \xrightarrow{\mathcal{C}} \quad -\bar{\psi}\gamma_\mu\gamma^5\psi = -A_\mu\end{aligned}$$

b) Expecting an analogy with the electromagnetic current vector, we will check the transformation properties of each.

	agree?		agree?		agree?
$J^\mu \xrightarrow{\mathcal{P}} J_\mu$		$J^\mu \xrightarrow{\mathcal{C}} -J^\mu$		$J^\mu \xrightarrow{\mathcal{CP}} -J_\mu$	
$V^\mu \xrightarrow{\mathcal{P}} V_\mu$	*yes*	$V^\mu \xrightarrow{\mathcal{C}} -V^\mu$	*yes*	$V^\mu \xrightarrow{\mathcal{CP}} -V_\mu$	*yes*
$A^\mu \xrightarrow{\mathcal{P}} -A_\mu$	*no*	$A^\mu \xrightarrow{\mathcal{C}} A^\mu$	*no*	$A^\mu \xrightarrow{\mathcal{CP}} -A_\mu$	*yes*

c) We will demonstrate that the weak Lagrangian,

$$\mathcal{L}_{\text{weak}} \approx \frac{G_F}{\sqrt{2}}(V_\mu - A_\mu)(V^\mu - A^\mu),$$

is not invariant under $\mathcal{C}$ or $\mathcal{P}$, yet is invariant under $\mathcal{CP}$.

Like before, I will directly compute all of the transformations using the table made above in part (*b*) above. First note that

$$\mathcal{L}_{\text{weak}} \propto V^2 - 2V_\mu A^\mu + A^2.$$

When we take each of the of transformations from above, we see that

$$\begin{aligned}V^2 - 2V_\mu A^\mu + A^2 &\xrightarrow{\mathcal{P}} V^2 + 2V_\mu A^\mu + A^2 \neq \mathcal{L}_{\text{weak}};\\ V^2 - 2V_\mu A^\mu + A^2 &\xrightarrow{\mathcal{C}} V^2 + 2V_\mu A^\mu + A^2 \neq \mathcal{L}_{\text{weak}};\\ V^2 - 2V_\mu A^\mu + A^2 &\xrightarrow{\mathcal{CP}} V^2 - 2V_\mu A^\mu + A^2 = \mathcal{L}_{\text{weak}}.\end{aligned}$$

So $\mathcal{L}_{\text{weak}}$ is not invariant under $\mathcal{C}$ or $\mathcal{P}$ by is under $\mathcal{CP}$, as we were required to demonstrate.

ὅπερ ἔδει δεῖξαι

3. Let us define the product of the 3 discrete symmetry transformations as $\Theta \equiv \mathcal{CPT}$. We must show that under Θ, the Dirac field transforms by the rule

$$\Theta\psi(x)\Theta^\dagger = \gamma^5\psi^*(-x),$$

where

$$\psi^*(x) \equiv (\psi(x)^\dagger)^\mathsf{T}.$$

Like so many times before, we will proceed by direct calculation.

$$\begin{aligned}
\Theta\psi(x)\Theta^\dagger &= \mathcal{CPT}\psi(t,\vec{x})\mathcal{T}^\dagger\mathcal{P}^\dagger\mathcal{C}^\dagger, \\
&= \mathcal{CP}\gamma^1\gamma^3\psi(-t,\vec{x})\mathcal{P}^\dagger\mathcal{C}^\dagger, \\
&= \eta_a\mathcal{C}\gamma^1\gamma^3\gamma^0\psi(-x)\mathcal{C}^\dagger, \\
&= -i\eta_a\gamma^1\gamma^3\gamma^0(\psi(-x)^\dagger\gamma^2)^\mathsf{T}, \\
&= -i\eta_a\gamma^1\gamma^3\gamma^0\gamma^{2\mathsf{T}}\psi^*(-x), \\
&= -i\eta_a\gamma^0\gamma^1\gamma^3\gamma^2\psi^*(-x), \\
&= i\eta_a\gamma^0\gamma^1\gamma^2\gamma^3\psi^*(-x), \\
&= \eta_a\gamma^5\psi^*(-x).
\end{aligned}$$

ὅπερ ἔδει δεῖξαι

4. For the following derivations it will be useful to recall that

$$\gamma^0_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i_W = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) We must show that any new matrices defined by

$$\gamma^\mu = U\gamma^\mu_W U^\dagger,$$

where U is an arbitrary 4×4 unitary matrix, satisfy the dirac algebra. This is proven by demonstrating that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

Knowing that the Weyl-representation γ^μ's satisfy the Dirac algebra, we will directly show that,

$$\begin{aligned}
\{\gamma^\mu,\gamma^\nu\} &= \{U\gamma^\mu_W U^\dagger, U\gamma^\nu_W U^\dagger\}, \\
&= U\gamma^\mu_W U^\dagger U\gamma^\nu_W U^\dagger + U\gamma^\nu_W U^\dagger U\gamma^\mu_W U^\dagger, \\
&= U(\gamma^\mu_W\gamma^\nu_W + \gamma^\nu_W\gamma^\mu_W)U^\dagger, \\
&= 2Ug^{\mu\nu}U^\dagger = 2g^{\mu\nu}.
\end{aligned}$$

ὅπερ ἔδει δεῖξαι

b) Consider the unitary matrix which produces the Dirac representation

$$U_D = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

We must show that U_D is in fact unitary and we must find the matrices γ^μ in the Dirac representation.

The unitarity of U_D is trivial

$$U_D U_D^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 1_{4\times 4}.$$

When the matrices are directly computed, we see that

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \gamma^i_W.$$

c) We must now show that in a general frame, the Dirac spinor takes the form,

$$u^s(p) = \begin{pmatrix} \sqrt{E+m}\xi^s \\ \vec{\sigma}\cdot\vec{p}\;\xi^s/\sqrt{E+m} \end{pmatrix}.$$

This is demonstrated by showing that it solves the Dirac equation, or, namely, that

$$\gamma^\mu p_\mu u^s(p) = m u^s(p).$$

This is simple to evaluate directly. Noting our Dirac representation of the γ^μ's and that $p_0 = E$, we see

$$\begin{aligned}
\gamma^\mu p_\mu u^s(p) &= \begin{pmatrix} p_0 & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p_0 \end{pmatrix}\begin{pmatrix} \sqrt{E+m}\xi^s \\ \vec{\sigma}\cdot\vec{p}\;\xi^s/\sqrt{E+m} \end{pmatrix}, \\
&= \begin{pmatrix} \left[E\sqrt{E+m} - \frac{E^2-m^2}{\sqrt{E+m}}\right]\xi^s \\ \left[\vec{\sigma}\cdot\vec{p}\sqrt{E+m} - \frac{E\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\right]\xi^s \end{pmatrix}, \\
&= \begin{pmatrix} \sqrt{E+m}\left(E - \frac{E^2-m^2}{E+m}\right)\xi^s \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}(E+m-E)\xi^s \end{pmatrix}, \\
&= \begin{pmatrix} \sqrt{E+m}(E-E+m)\xi^s \\ m\vec{\sigma}\cdot\vec{p}\;\xi^s/\sqrt{E+m} \end{pmatrix}, \\
&= \begin{pmatrix} m\sqrt{E+m}\;\xi^s \\ m\vec{\sigma}\cdot\vec{p}\;\xi^s/\sqrt{E+m} \end{pmatrix}, \\
&= m u^s(p).
\end{aligned}$$

ὅπερ ἔδει δεῖξαι

d) We must show that the solution found in part (*c*) is normalized in the standard way.

Given that ξ is normalized such that $\xi\xi^\dagger = 1$, we see that

$$\begin{aligned}
\bar{u}u = u^\dagger\gamma^0 u &= \begin{pmatrix} \sqrt{E+m}\;\xi^\dagger & \vec{\sigma}\cdot\vec{p}\;\xi^\dagger/\sqrt{E+m} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{E+m}\;\xi \\ \vec{\sigma}\cdot\vec{p}\;\xi/\sqrt{E+m} \end{pmatrix}, \\
&= \xi^\dagger\xi\left((E+m) - \frac{(\vec{\sigma}\cdot\vec{p})^2}{E+m}\right), \\
&= \frac{E^2+2mE+m^2-\vec{p}^2}{E+m}, \\
&= \frac{E^2+2mE+m^2-E^2+m^2}{E+m}, \\
&= \frac{2mE+2m^2}{E+m}, \\
&= 2m.
\end{aligned}$$

ὅπερ ἔδει δεῖξαι